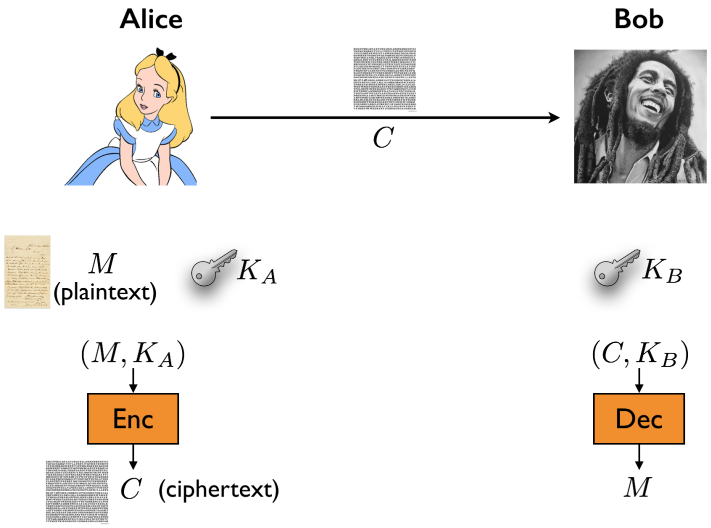
Cryptography

1 Private-key cryptographic system

The general setting is as follows. There are three parties named Alice, Bob and Eve. Alice's goal is to send a private message to Bob over some channel. In particular, Alice wants only Bob to know what her message is, but unfortunately, any message she sends over the channel can be intercepted by the eavesdropper Eve. Therefore, she would like to first encrypt her message (which is also known as the *plaintext*) and then send the encrypted message (the *ciphertext*) to Bob over the channel. Bob should be able to decrypt the ciphertext and recover the plaintext. If the system is secure, then Eve learns no information about the plaintext by seeing the ciphertext. If the system is not secure, we will call it broken. We assume that Eve knows the encryption and decryption algorithms used by Alice and Bob respectively.

In a private-key cryptographic system, a protocol is designed as follows. We assume Alice has a private key K_A and Bob has a private key K_B. These keys help Alice and Bob encrypt and decrypt messages. In particular, if M is the plaintext that Alice wants to send, she encrypts it using an encryption algorithm Enc that takes M and K_A as input, and produces a ciphertext C as output: $\text{Enc}(M, K_A) = C$. This ciphertext C is sent to Bob, and Bob uses a decryption algorithm Dec that takes a ciphertext C and his private key K_B as input, and produces a plaintext M: $\text{Dec}(C, K_B) = M$. (We assume that M, C, K_A, K_B are encoded as strings over some finite alphabet Σ.)



A very simple example of a private-key protocol is the well-known Caesar shift.[1] In this protocol, $0 \leq K_A = K_B \leq 25$ and a plaintext is encrypted by replacing each letter with a letter which is K_A positions down the alphabet. This system is easy to break since the number of possibilities for the key is very small. Therefore one can try each possible key value one by one.

There are much more sophisticated private-key protocols. We now present a simple one that is provably perfectly secure.

1.1 One-time pad

In this section, we assume that the plaintext $M \in \{0,1\}^n$ is a binary string of length n. Then we choose $K_A = K_B \in \{0,1\}^n$ uniformly at random (and denote it by K). The encryption algorithm takes the bit-wise xor of M and K to produce an n-bit ciphertext C. Or in other words, for each i, the i'th bit of C, $C[i]$, is defined to be $M[i] \oplus K[i]$. Below is an example.

Encryption:

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    M =  01011010111010100000111
⊕   K =  11001100010101111000101
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    C =  10010110101111011000010
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The decryption algorithm is exactly the same as the encryption algorithm. It takes C and K as input, and produces M by taking the bit-wise xor of C and K. Observe that for all i,

$$C[i] \oplus K[i] = (M[i] \oplus K[i]) \oplus K[i] = M[i] \oplus (K[i] \oplus K[i]) = M[i],$$

so the decryption algorithm correctly recovers the original message M.

For any plaintext $M \in \{0,1\}^n$, if $K \in \{0,1\}^n$ is chosen uniformly at random, then the ciphertext C is a uniformly random element of $\{0,1\}^n$. This means that Eve learns nothing about M by seeing C, so the system is perfectly secure. The downside is that Alice and Bob have to share a key that is as long as the message itself.[2]

Exercise (One-time pad based on modular multiplication)**.** Note that in the one-time pad cryptographic scheme described above, the underlying universe that we are dealing

[1] See https://en.wikipedia.org/wiki/Caesar_cipher for details on the Caesar shift.

[2] For the system to remain perfectly secure, you should not reuse the same key for more than one message. This is the reason for the name "one-time pad".

with is $\mathbb{Z}_2^n$ with the operation being bit-wise xor. Describe a similar scheme that uses $\mathbb{Z}_N^*$ as the universe.

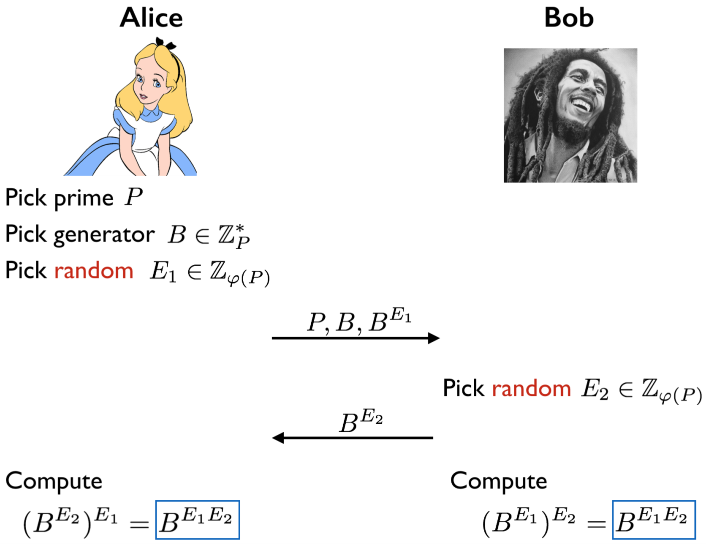
It is natural to ask whether there is any perfectly secure system like one-time pad that uses a shorter key. Claude Shannon proved that the answer is "no". To state his result informally, he showed that if K is shorter than M and Eve is computationally unbounded, then Eve can learn some information about the message M.

Given this, we let computational complexity come to our rescue. It is completely reasonable to assume that Eve is indeed computationally *bounded*. So from now on, we will assume that Eve is a polynomial-time agent.

1.2 Diffie-Hellman secret key exchange

We present a protocol for Alice and Bob to agree on a secret key K by communicating publicly. The protocol is secure if Eve has no information about K even though she sees all the communication between Alice and Bob. This sounds like an impossible task, but it is actually believed to be feasible.

The protocol makes use of the assumption that the Discrete Log problem is computationally hard and it goes as follows. Alice picks privately a (sufficiently large) random prime number P, a generator B in $\mathbb{Z}_P^*$, and a random exponent $E_1 \in \mathbb{Z}_{\varphi(P)}$.[3] She computes B^{E_1} in $\mathbb{Z}_P^*$ and sends over to Bob P, B, B^{E_1}. Bob privately picks a random exponent $E_2 \in \mathbb{Z}_{\varphi(P)}$ and computes B^{E_2} in $\mathbb{Z}_P^*$. He sends B^{E_2} to Alice. At this point both players can privately compute $S = B^{E_1 E_2}$ in $\mathbb{Z}_P^*$, which is defined as the secret key that they now share. The protocol is illustrated below.


Alice
Bob
Pick prime P
Pick generator $B \in \mathbb{Z}_P^*$
Pick random $E_1 \in \mathbb{Z}_{\varphi(P)}$
P, B, B^{E_1}
Pick random $E_2 \in \mathbb{Z}_{\varphi(P)}$
B^{E_2}
Compute
$(B^{E_2})^{E_1} = \boxed{B^{E_1 E_2}}$
Compute
$(B^{E_1})^{E_2} = \boxed{B^{E_1 E_2}}$

There are two important questions related to this protocol. First, are all the operations done by Alice and Bob polynomial-time computable? Second, how secure is the system?

Even though we won't explicitly discuss it, every computation done by Alice and Bob can indeed be done in polynomial time. For the security, observe that we definitely need the Discrete Log problem to be computationally hard, because otherwise, Eve can compute E_1 from B^{E_1} and E_2 from B^{E_2}. Then it is easy for her to compute the "secret" $B^{E_1 E_2}$ (since she also knows B). To be more careful though, we want that given P, B, B^{E_1}, B^{E_2} (i.e. what Eve sees), it is computationally hard to compute $B^{E_1 E_2}$. This is known as the Diffie-Hellman assumption. Unfortunately we cannot prove this assumption since if we could, we would be also proving $\mathsf{P} \neq \mathsf{NP}$. Even if the Diffie-Hellman assumption holds, we should not be satisfied. Not only we don't want Eve to compute the secret $B^{E_1 E_2}$, but we don't want her to gain *any* information about $B^{E_1 E_2}$ (e.g. not

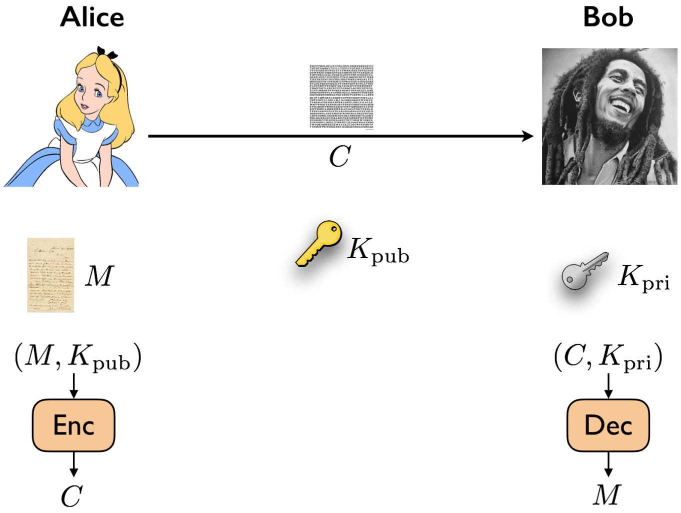
[3] Why is the exponent chosen from $\mathbb{Z}_{\varphi(P)}$? Recall that thanks to Euler's Theorem, if we are exponentiating an element $A \in \mathbb{Z}_N^*$, then we can effectively think of the exponent as living in the set $\mathbb{Z}_{\varphi(N)}$.

even the first bit of it). The assumption that Eve learns nothing about the secret is known as the Decisional Diffie-Hellman assumption.[4]

2 Public-key cryptographic system

In a public-key cryptographic system, our goal is to design a protocol that allows Alice to send a message to Bob without the need of having to exchange messages in order to share a secret key.

In order to establish this, a public-key cryptographic system uses the following general strategy. Bob generates a tuple of keys $(K_{\text{pri}}, K_{\text{pub}})$, where K_{pri} is called the *private key* and is kept private to him, and K_{pub} is called the *public key* and is published to the world. If someone (e.g. Alice) wants to send a message to Bob, they use the public key to encrypt their message M. That is, the ciphertext C is produced by running an encryption algorithm Enc(M, K_{pub}). Once Bob receives C, he decrypts it using his private key by running a decryption algorithm Dec(C, K_{pri}).



We now present different instantiations of this idea.

2.1 ElGamal public-key cryptographic system

The first public-key protocol we present is similar in nature to the Diffie-Hellman secret-key exchange protocol. In fact, it is basically combining the Diffie-Hellman secret-key exchange protocol with the one-time pad protocol. It is easy to combine them to create a private-key cryptographic system. Here, we'll see that they can also be combined to create a public-key cryptographic system.

To make the correspondence clear with the Diffie-Hellman secret-key exchange protocol, we reverse the roles of Alice and Bob. Below are the details.

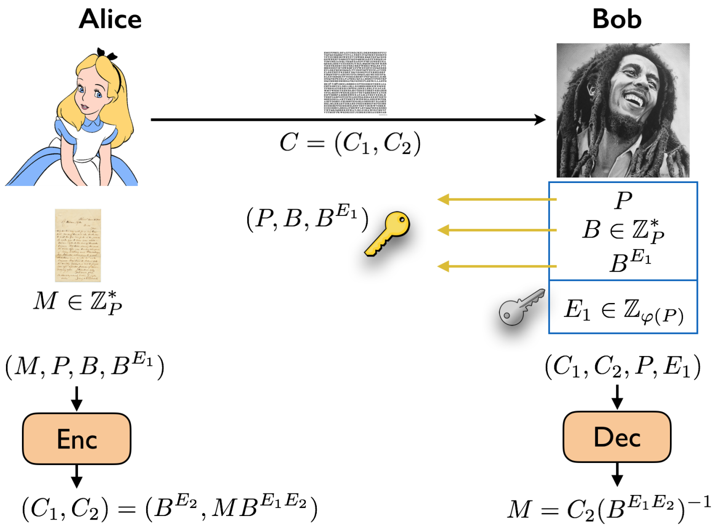
In the ElGamal protcol, Bob picks a (sufficiently large) prime number P, a generator $B \in \mathbb{Z}_P^*$, and a random exponent $E_1 \in \mathbb{Z}_{\varphi(P)}$. He computes B^{E_1} in $\mathbb{Z}_P^*$. The private key is $K_{\text{pri}} = E_1$ and the public key is $K_{\text{pub}} = (P, B, B^{E_1})$.

The message M that Alice wants to send is viewed as an element of $\mathbb{Z}_P^*$ (it is easy to agree on an encoding scheme to do this). To encrypt her message, Alice does the following. She picks a random exponent $E_2 \in \mathbb{Z}_{\varphi(P)}$, and computes B^{E_2}, $B^{E_1 E_2}$ and $MB^{E_1 E_2}$ in $\mathbb{Z}_P^*$. Then the ciphertext she sends over to Bob is $C = (C_1, C_2) = (B^{E_2}, MB^{E_1 E_2})$. Once Bob receives this message, using C_1 he first computes $C_1^{E_1} = B^{E_1 E_2}$ in $\mathbb{Z}_P^*$. Note

[4] Decisional Diffie-Hellman assumption turns out to be false in the group $\mathbb{Z}_P^*$, but there are other cyclic groups for which experts believe the assumption should hold.

that this is the *secret key* from the Diffie-Hellman protocol. Let's call it S. He computes S^{-1} in $\mathbb{Z}_P^*$. Then he computes $C_2 S^{-1} = M$ to recover the original message M.

Even though this may seem like a non-simple protocol, it has a simple summary once you understand the Diffie-Hellman secret-key exchange protocol. Effectively, Alice and Bob are sharing the private secret $S = B^{E_1 E_2}$ as in the Diffie-Hellman secret-key exchange protocol. Alice "masks" her message M using S (note that this is equivalent to doing a one-time pad in the universe $\mathbb{Z}_P^*$). Since Bob also knows S, he can compute its inverse, and therefore recover M.



As in the Diffie-Hellman secret-key exchange protocol, all the computation done by Alice and Bob can be carried out in polynomial time. Furthermore, the security is based on the same assumptions as in the Diffie-Hellman secret-key exchange protocol. The details are omitted.

2.2 RSA public-key cryptographic system

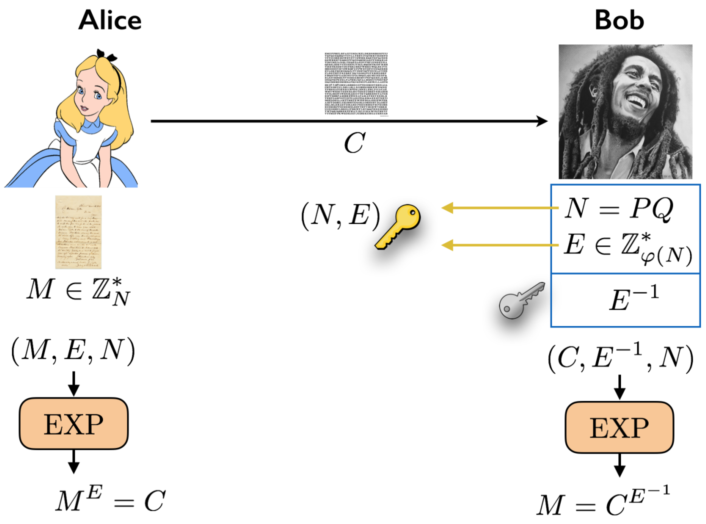
The RSA cryptographic system uses the assumption that taking roots in the modular universe is a computationally hard problem. Notice that taking roots is the inverse of the exponentiation function. And in RSA, the encryption is indeed done using the exponentiation function. Below are the details.

First, Bob picks two (sufficiently large) distinct prime numbers P and Q. He multiplies them together to obtain $N = PQ$.[5] He picks an exponent $E \in \mathbb{Z}_{\varphi(N)}^*$.[6] He computes E^{-1} in $\mathbb{Z}_{\varphi(N)}^*$ and keeps that as his private key, $K_{\text{pri}} = E^{-1}$. The public key is $K_{\text{pub}} = (N, E)$.

The message M that Alice wants to send to Bob is viewed as an element of $\mathbb{Z}_N^*$. To encrypt her message, she computes $C = M^E$ in $\mathbb{Z}_N^*$, and sends it over to Bob. The decryption algorithm happens to be exactly the same as the encryption algorithm. Once Bob receives C, he computes $C^{E^{-1}}$ in $\mathbb{Z}_N^*$, and recovers M since $C^{E^{-1}} = (M^E)^{E^{-1}} = M^{EE^{-1}} = M$.

[5] Why is N chosen to be a product of primes and not just a prime number? We will explore this question after we describe the protocol.

[6] Why is the exponent chosen from $\mathbb{Z}_{\varphi(N)}^*$? Once again, if we are exponentiating an element $A \in \mathbb{Z}_N^*$, then we can effectively think of the exponent as living in the set $\mathbb{Z}_{\varphi(N)}$. In the RSA protocol, we will need the exponent to have an inverse in $\mathbb{Z}_{\varphi(N)}$ and therefore we pick it from $\mathbb{Z}_{\varphi(N)}^*$.



As before, it can be shown that all the computation done by Alice and Bob is polynomial-time. We now make a few comments about the security of the system. What is the advantage that Bob has over Eve that allows him to decrypt the message? If Eve could compute E^{-1} herself, then she would be able to decrypt the message as well. To compute E^{-1}, you need to know $\varphi(N)$ since E^{-1} lives in $\mathbb{Z}^*_{\varphi(N)}$. Bob's advantage is that he can easily compute $\varphi(N)$ because $\varphi(N) = (P - 1)(Q - 1)$. In other words, the advantage that Bob has is that he knows the prime factorization of N. If Eve could factor N efficiently, then she could also easily compute $\varphi(N) = (P - 1)(Q - 1)$. It turns out that factoring N and computing $\varphi(N)$ are computationally equivalent in the following sense. Clearly, as we argued, if we can factor N in polynomial time, then we can compute $\varphi(N)$ in polynomial time. Furthermore, if we can compute $\varphi(N)$ in polynomial time, then we can factor N in polynomial time (we leave this as an exercise to the reader).

One might ask if computing $\varphi(N)$ is the only way to crack RSA. We don't know the answer to this question. So we cannot rule out that there might be some other devious way of recovering the message M without computing $\varphi(N)$ (or factoring N).

3 Check Your Understanding

Problem. 1. Consider the one-time pad cryptographic system. Show that for any plaintext $M \in \{0, 1\}^n$, if the key $K \in \{0, 1\}^n$ is chosen uniformly at random, then the ciphertext C is a uniformly random element of $\{0, 1\}^n$.

2. Describe the Diffie-Hellman secret key exchange protocol.

3. Quantum computers can compute the discrete log problem efficiently. Explain how quantum computers can be used to break the Diffie-Hellman secret key exchange protocol.

4. Describe the RSA public-key protocol.

5. Quantum computers can compute the discrete root problem efficiently. Explain how quantum computers can be used to break the RSA public-key protocol.

6. Quantum computers can compute the factoring problem (given an integer N, output its prime factors) efficiently. Explain how quantum computers can be used to break the RSA public-key protocol.

4 High-Order Bits

Important.

1. Understand the general outline of how private-key protocols and public-key protocols work, independent of specific implementations of these ideas.
2. Make sure you understand how the one-time pad private-key protocol works and why it is a perfectly secure protocol.
3. You should be able to describe the Diffie-Hellman secret key exchange protocol and how it relates to the assumed hardness of the discrete log problem.
4. You should be able to describe the RSA public-key protocol and how it relates to the assumed hardness of the discrete root problem. You should also understand the relevance of the factoring problem in this setting.